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① $\int x\sqrt{x^2+2}\,dx$

$t=\sqrt{x^2+2}$ とおくと, $dt=\frac{2x}{2\sqrt{x^2+2}}dx \Leftrightarrow \frac{\sqrt{x^2+2}}{x}dt=dx$

(与式)$=\int x\sqrt{x^2+2}\,\frac{\sqrt{x^2+2}}{x}dt = \int t^2dt = \frac{1}{3}t^3+C$

$=\boxed{\frac{1}{3}(x^2+2)\sqrt{x^2+2}+C}$

[別解] $\frac{1}{2}\int(x^2+2)'(x^2+2)^{\frac{1}{2}}dx=\frac{1}{2}\left\{\frac{2}{3}(x^2+2)^{\frac{3}{2}}\right\}+C$ (以下略)

② $\int x(1+x^2)\sqrt{1+x^2}\,dx$

$t=\sqrt{1+x^2}$ とおくと, $dt=\frac{2x}{2\sqrt{1+x^2}}dx\Leftrightarrow \frac{\sqrt{1+x^2}}{x}dt=dx$

(与式)$=\int x\cdot t^2\cdot t\cdot\frac{\sqrt{1+x^2}}{x}dt=\int t^4dt=\frac{1}{5}t^5+C$

$=\boxed{\frac{1}{5}(1+x^2)^2\sqrt{1+x^2}+C}$

[別解] $\frac{1}{2}\int(1+x^2)'(1+x^2)^{\frac{3}{2}}dx=\frac{1}{2}\left\{\frac{2}{5}(1+x^2)^{\frac{5}{2}}\right\}+C$ (以下略)

③ $\int\frac{x}{\sqrt{x+1}}dx$

$t=\sqrt{x+1}$ とおくと, $dt=\frac{dx}{2\sqrt{x+1}} \Leftrightarrow 2tdt=dx$

(与式)$=\int\frac{t^2-1}{t}\times 2tdt=2\left(\frac{t^3}{3}-t\right)+C$

$=\boxed{\frac{2}{3}(x-2)\sqrt{x+1}+C}$

④ $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ $t=\sqrt{x}$ とおくと, $dt=\frac{1}{2\sqrt{x}}dx$

(与式)$=\int e^t\cdot 2dt = 2e^t+C = \boxed{2e^{\sqrt{x}}+C}$

⑤ $\int\frac{1}{e^x+1}dx$ $t=e^x+1$ とおくと, $dt=e^xdx\Leftrightarrow dt=(t-1)dx$

(与式)$=\int\frac{1}{t(t-1)}dt$ (《類題1》(12) !!)

$=\log\left|\frac{t-1}{t}\right|+C = \boxed{\log\left(\frac{e^x}{e^x+1}\right)+C}$

[別解] 上下に e^{-x} をかけて, $\frac{f'}{f}$ を利用

⑥ $\int\frac{1}{x(\log x)^2}dx$ $t=\log x$ とおくと, $dt=\frac{dx}{x}$

(与式)$=\int t^{-2}dt = -t^{-1}+C = \boxed{-\frac{1}{\log x}+C}$

[別解] $\int(\log x)'(\log x)^{-2}dx=-(\log x)^{-1}+C$

⑦ $\int\sin^3x\cos x\,dx$ $t=\sin x$ とおくと, $dt=\cos x\,dx$

(与式)$=\int t^3dt=\frac{1}{4}t^4+C=\boxed{\frac{1}{4}\sin^4x+C}$

[別解] $\int(\sin x)'(\sin x)^3dx=\frac{1}{4}(\sin x)^4+C$

⑧ $\int\cos^3x\,dx$ $t=\sin x$ とおくと, $dt=\cos x\,dx$

(与式)$=\int(1-\sin^2x)\cos x\,dx=\int(1-t^2)dt = t-\frac{1}{3}t^3+C$

$=\boxed{\sin x-\frac{1}{3}\sin^3x+C}$

⑨ $\int\sin^5x\,dx$ $t=\cos x$ とおくと, $dt=-\sin x\,dx$

(与式)$=\int(1-\cos^2x)^2\sin x\,dx=\int(1-t^2)^2(-dt)=-\int(t^4-2t^2+1)dt$

$=-\frac{1}{5}t^5+\frac{2}{3}t^3-t+C=\boxed{-\frac{1}{5}\cos^5x+\frac{2}{3}\cos^3x-\cos x+C}$

⑩ $\int\tan^3x\,dx$ $t=\tan x$ とおくと,

$dt=\frac{1}{\cos^2x}dx=(1+\tan^2x)dx$ (∵三角比の相互関係)

(与式)$=\int\frac{t^3}{1+t^2}dt=\int\left(t-\frac{t}{t^2+1}\right)dt$ (∵次数下げ)

$=\frac{t^2}{2}-\frac{1}{2}\log(t^2+1)+C = \boxed{\frac{1}{2}\tan^2x-\frac{1}{2}\log(1+\tan^2x)+C}$

$=\boxed{\frac{1}{2}\tan^2x+\frac{1}{2}\log(\cos^2x)+C}$

$=\boxed{\frac{1}{2}\tan^2x+\log|\cos x|+C}$

《類題4》

(1) [やり方1] $t=x^2+2$ と置換

[やり方2] $\int(x^2+2)'(x^2+2)^2dx=\boxed{\frac{1}{3}(x^2+2)^3+C}$

(2) [やり方1] $t=4-x^2$ と置換

[やり方2] $\int x(4-x^2)^{-\frac{1}{2}}dx=-\frac{1}{2}\int(4-x^2)'(4-x^2)^{-\frac{1}{2}}dx$

$=-\frac{1}{2}\left\{2(4-x^2)^{\frac{1}{2}}\right\}+C=\boxed{-\sqrt{4-x^2}+C}$

(3) $t=x+2$ とおくと, $dt=dx$ $(x=t-2)$

(与式) $=\int(2t-3)\sqrt{t}\,dt=\int\left(2t^{\frac{3}{2}}-3t^{\frac{1}{2}}\right)dt=\frac{4}{5}t^{\frac{5}{2}}-2t^{\frac{3}{2}}+C$

$=\boxed{\frac{4}{5}(x+2)^2\sqrt{x+2}-2(x+2)\sqrt{x+2}+C}$

(4) $t=x+1$ とおくと, $dt=dx$ $(x=t-1)$

(与式) $=\int\frac{t-1}{t^2}dt=\int\left(\frac{1}{t}-t^{-2}\right)dt=\log|t|+t^{-1}+C$

$=\boxed{\log|x+1|+\frac{1}{x+1}+C}$

(5) $t=1+\sqrt{x}$ とおくと, $dt=\frac{1}{2\sqrt{x}}dx \Leftrightarrow 2\sqrt{x}\,dt=dx$

(与式) $=\int\frac{t-1}{t}\cdot2(t-1)dt=2\int\left(t-2+\frac{1}{t}\right)dt=t^2-4t+2\log|t|+C$

$=(1+2\sqrt{x}+x)-4(1+\sqrt{x})+2\log|1+\sqrt{2}|+C$

$=\boxed{x-2\sqrt{x}+2\log(1+\sqrt{x})-3+C}$

(※ $t=\sqrt{x}$ とおくと,『-3』が無くなるけど, これでも〇。Cに吸い込まれたのだ。)

(6) $t=e^{-x}$ とおくと, $dt=-e^{-x}dx$

(与式) $=\int\frac{t^2}{1+t}\cdot\frac{dt}{-t}=\int\left(\frac{1}{t+1}-1\right)dt=\log|t+1|-t+C$

$=\boxed{\log(e^{-x}+1)-e^{-x}+C}$

(7) [やり方1] $t=x^2$ と置換 [やり方2] $\frac{1}{6}\int(3x^2)'e^{3x^2}dx=\boxed{\frac{1}{6}e^{3x^2}+C}$

(8) $t=\log x-1$ とおくと, $dt=\frac{1}{x}dx$

(与式) $=\int\frac{t+1}{x\cdot t^2}xdt=\int\left(\frac{1}{t}+t^{-2}\right)dt=\log|t|-t^{-1}+C$

$=\boxed{\log|\log x-1|-\frac{1}{\log x-1}+C}$